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Probability Theory and Applications (MA208)
Problem Sheet - 7

Further Characterizations of Random Variables

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1. Find the expected value of the following random variables.
   - (a) The random variable X defined in Problem 4.1.
   - (b) The random variable X defined in Problem 4.2.
   - (c) The random variable T defined in Problem 4.6.
   - (d) The random variable X defined in Problem 4.18.
2. Show that $E(X)$ does not exist for the random variable X defined in Problem 4.25.
3. The following represents the probability distribution of D, the daily demand of a certain product. Evaluate $E(D)$.

$$d : 1, 2, 3, 4, 5,$$
$$P(D = d) : 0.1, 0.1, 0.3, 0.3, 0.2.$$

4. In the manufacture of petroleum, the distilling temperature, say T (degrees centigrade), is crucial in determining the quality of the final product. Suppose that T is considered as a random variable uniformly distributed over $(150, 300)$.
   Suppose that it costs C_1 dollars to produce one gallon of petroleum. If the oil distills at a temperature less than $200°C$, the product is known as naphtha and sells for C_2 dollars per gallon. If it is distilled at a temperature greater than $200°C$, it is known as refined oil distillate and sells for C_3 dollars per gallon. Find the expected net profit (per gallon).
5. A certain alloy is formed by combining the melted mixture of two metals. The resulting alloy contains a certain percent of lead, say X, which may be considered as a random variable. Suppose that X has the following pdf:

$$f(x) = \frac{3}{5} 10^{-5} x(100 - x), \quad 0 \leq x \leq 100.$$

   Suppose that P, the net profit realized in selling this alloy (per pound), is the following function of the percent content of lead: $P = C_1 + C_2 X$. Compute the expected profit (per pound).
6. Suppose that an electronic device has a life length X (in units of 1000 hours) which is considered as a continuous random variable with the following pdf:

$$f(x) = e^{-x}, \quad x > 0.$$

   Suppose that the cost of manufacturing one such item is \$2.00. The manufacturer sells the item for \$5.00, but guarantees a total refund if $X \leq 0.9$. What is the manufacturer's expected profit per item?

7. The first 5 repetitions of an experiment cost \$10 each. All subsequent repetitions cost \$5 each. Suppose that the experiment is repeated until the first successful outcome occurs. If the probability of a successful outcome always equals 0.9, and if the repetitions are independent, what is the expected cost of the entire operation?

8. A lot is known to contain 2 defective and 8 nondefective items. If these items are inspected at random, one after another, what is the expected number of items that must be chosen *for inspection* in order to remove all the defective ones?

9. A lot of 10 electric motors must either be totally rejected or is sold, depending on the outcome of the following procedure: Two motors are chosen at random and inspected. If one or more are defective, the lot is rejected. Otherwise it is accepted. Suppose that each motor costs \$75 and is sold for \$100. If the lot contains 1 defective motor, what is the manufacturer's expected profit?

10. Suppose that D, the daily demand for an item, is a random variable with the following probability distribution:

$$P(D = d) = C2^d/d!, \quad d = 1, 2, 3, 4.$$

    (a) Evaluate the constant C.
    (b) Compute the expected demand.
    (c) Suppose that an item is sold for \$5.00. A manufacturer produces K items daily. Any item which is not sold at the end of the day must be discarded at a loss of \$3.00.
        (i) Find the probability distribution of the daily profit, as a function of K.
        (ii) How many items should be manufactured to maximize the expected daily profit?

11. (a) With $N = 50$, $p = 0.3$, perform some computations to find that value of k which minimizes $E(X)$ in Example 7.12.
    (b) Using the above values of N and p and using $k = 5, 10, 25$, determine for each of these values of k whether "group testing" is preferable.

12. Suppose that X and Y are independent random variables with the following pdf's:

$$f(x) = 8/x^3, \quad x > 2; \quad g(y) = 2y, \quad 0 < y < 1.$$

    (a) Find the pdf of $Z = XY$.
    (a) Obtain $E(Z)$ in two ays:
        (i) using the pdf of Z as obtained in (a).
        (ii) Directly, without using the pdf of Z.

13. Suppose that X has pdf

$$f(x) = 8/x^3, \quad x > 2.$$

    Let $W = \frac{1}{3}X$.
    (a) Evaluate $E(W)$ using the pdf of W.
    (b) Evaluate $E(W)$ without using the pdf of W.

14. A fair die is tossed 72 times. Given that X is the number of times six appears, evaluate $E(X^2)$.

15. Find the expected value and variance of the random variables Y and Z of Problem 5.2.

16. Find the expected value and variance of the random variable Y of Problem 5.3.

17. Find the expected value and variance of the random variables Y and Z of Problem 5.5.

18. Find the expected value and variance of the random variables Y, Z, and W of Problem 5.6.

19. Find the expected value and variance of the random variables V and S of Problem 5.7.

20. Find the expected value and variance of the random variable Y of Problem 5.10 for each of the three cases.

21. Find the expected value and variance of the random variable A of Problem 6.7.

22. Find the expected value and variance of the random variable H of Problem 6.11.

23. Find the expected value and variance of the random variable W of Problem 6.13.

24. Suppose that X is a random variable for which $E(X) = 10$ and $V(X) = 25$. For what positive values of a and b does $Y = aX - b$ have expectation 0 and variance 1?

25. Suppose that S, a random voltage, varies between 0 and 1 volt and is uniformly distributed over that interval. Suppose that the signal S is perturbed by an additive, independent random noise N which is uniformly distributed between 0 and 2 volts.
    (a) Find the expected voltage of the signal, taking noise into account.
    (b) Find the expected power when the perturbed signal is applied to a resistor of 2 ohms.

26. Suppose that X is uniformly distributed over $[-a, 3a]$. Find the variance of X.

27. A target is made of three concentric circles of radii $1/\sqrt{3}, 1$, and $\sqrt{3}$ feet. Shots within the inner circle count 4 points, within the next ring 3 points, and within the third ring 2 points. Shots outside the target count zero. Let R be the random variable representing the distance of the hit from the center. Suppose that the pdf of R is $f(r) = 2/\pi(1+r^2)$, $r > 0$. Compute the expected value of the score after 5 shots.

28. Suppose that the continuous random variable X has pdf

$$f(x) = 2xe^{-x^2}, \quad x \geq 0.$$

Let $Y = X^2$. Evaluate $E(Y)$:
    (a) directly without first obtaining the pdf of Y,
    (b) by first obtaining the pdf of Y.

29. Suppose that the two-dimensional random variable (X, Y) is uniformly distributed over the triangle in Fig. 7.15. Evaluate $V(X)$ and $V(Y)$.

30. Suppose that (X, Y) is uniformly distributed over the triangle in Fig. 7.16.
    (a) Obtain the marginal pdf of X and of Y.
    (b) Evaluate $V(X)$ and $V(Y)$.

FIGURE 7.15
FIGURE 7.16

31. Suppose that X and Y are random variables for which $E(X) = \mu_x$, $E(Y) = \mu_y$, $V(X) = \sigma_x^2$, and $V(Y) = \sigma_y^2$. Using Theorem 7.7, obtain an approximation for $E(Z)$ and $V(Z)$, where $Z = X/Y$.

32. Suppose that X and Y are independent random variables, each uniformly distributed over $(1, 2)$. Let $Z = X/Y$.

    (a) Using Theorem 7.7, obtain approximate expressions for $E(Z)$ and $V(Z)$.

    (b) Using Theorem 6.5, obtain the pdf of Z and then find the exact value of $E(Z)$ and $V(Z)$. Compare with (a).

33. Show that if X is a continuous random variable with pdf f having the property that the graph of f is symmetric about $x = a$, then $E(X) = a$, provided that $E(X)$ exists. (See Example 7.16.)

34. (a) Suppose that the random variable X assumes the values -1 and 1 each with probability $\frac{1}{2}$. Consider $P[|X - E(X)| \geq k\sqrt{V(X)}]$ as a function of k, $k > 0$. Plot this function of k and, on the same coordinate system, plot the upper bound of the above probability as given by Chebyshev's inequality.

    (b) Same as (a) except that $P(X = -1) = \frac{1}{3}$, $P(X = 1) = \frac{2}{3}$.

35. Compare the upper bound on the probability $P[|X - E(X)| \geq 2\sqrt{V(X)}]$ obtained from Chebyshev's inequality with the exact probability if X is uniformly distributed over $(-1, 3)$.

36. Verify Eq. (7.17).

37. Suppose that the two-dimensional random variable (X, Y) is uniformly distributed over R, where R is defined by $\{(x, y) | x^2 + y^2 \leq 1, y \geq 0\}$. (See Fig. 7.17.) Evaluate ρ_{xy}, the correlation coefficient:

    FIGURE 7.17 FIGURE 7.18

38. Suppose that the two-dimensional random variable (X, Y) has pdf given by

$$\begin{aligned} f(x, y) &= ke^{-y}, & 0 < x < y < 1 \\ &= 0, & \text{elsewhere.} \end{aligned}$$

    (See Fig. 7.18.) Find the correlation coefficient ρ_{xy}.

39. The following example illustrates that $\rho = 0$ does not imply independence. Suppose that (X, Y) has a joint probability distribution given by Table 7.1.

    (a) Show that $E(XY) = E(X)E(Y)$ and hence $\rho = 0$.

    (b) Indicate why X and Y are not independent.

    (c) Show that this example may be generalized as follows. The choice of the number $\frac{1}{8}$ is not crucial. What is important is that all the circled values are the same, all the boxed values are the same, and the center value equals zero.

    TABLE 7.1

X Y	-1	0	1
-1	$\enclose{circle}{\frac{1}{8}}$	$\boxed{\frac{1}{8}}$	$\enclose{circle}{\frac{1}{8}}$
0	$\boxed{\frac{1}{8}}$	0	$\boxed{\frac{1}{8}}$
1	$\enclose{circle}{\frac{1}{8}}$	$\boxed{\frac{1}{8}}$	$\enclose{circle}{\frac{1}{8}}$

40. Suppose that A and B are two events associated with an experiment ϵ. Suppose that $P(A) > 0$ and $P(B) > 0$. Let the random variables X and Y be defined as follows.

$$X = 1 \text{ if } A \text{ occurs and } 0 \text{ otherwise,}$$
$$Y = 1 \text{ if } B \text{ occurs and } 0 \text{ otherwise.}$$

Show that $\rho_{xy} = 0$ implies that X and Y are independent.

41. Prove Theorem 7.14.

42. For the random variable (X, Y) defined in Problem 6.15, evaluate $E(X|y)$, $E(Y|x)$, and check that $E(X) = E[E(X|Y)]$ and $E(Y) = E[E(Y|X)]$.

43. Prove Theorem 7.16.

44. Prove Theorem 7.17. [*Hint:* For the continuous case, multiply the equation $E(Y|x) = Ax + B$ by $g(x)$, the pdf of X, and integrate from $-\infty$ to ∞. Do the same thing, using $xg(x)$ and then solve the resulting two equations for A and for B.]

45. Prove Theorem 7.18.

46. If X, Y, and Z are uncorrelated random variables with standard deviations $5, 12$, and 9, respectively and if $U = X + Y$ and $V = Y + Z$, evaluate the correlation coefficient between U and V.

47. Suppose that both of the regression curves of the mean are in fact linear. Specifically, assume that $E(Y|x) = -\frac{3}{2}x - 2$ and $E(X|y) = -\frac{3}{5}y - 3$.
    (a) Determine the correlation coefficient ρ.
    (b) Determine $E(X)$ and $E(Y)$.

48. Consider weather forecasting with two alternatives: “rain” or “no rain” in the next 24 hours. Suppose that p = Prob(rain in next 24 hours) $> 1/2$. The forecaster scores 1 point if he is correct and 0 points if not. In making n forecasts, a forecaster with no ability whatsoever chooses at random r days $(0 \leq r \leq n)$ to say “rain” and the remaining $n - r$ days to say “no rain.” His total point score is S_n. Compute $E(S_n)$ and $Var(S_n)$ and find that value of r for which $E(S_n)$ is largest. [*Hint:* Let $X_i = 1$ or 0 depending on whether the ith forecast is correct or not. Then $S_n = \sum_{i=1}^{n} X_i$. Note that the X_i’s are *not* independent.]

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